STABILITY OF GAS-BUBBLE EQUILIBRIUM SHAPE IN UNIFORM FLOW OF AN IDEAL FLUID

A. G. Petrov

UDC 532.529

Steady-state motion of a bubble in the shape of an ellipsoid of revolution has been studied [1, 2]. Steady-state motion and small oscillations of an ellipsoid of revolution around the equilibrium state were studied with the help of Lagrangian equations [3]. In this paper, possible equilibrium shapes of a bubble in the form of a triaxial ellipsoid are studied. The dependence of the pressure difference at the stagnation point and within the gas bubble on deformation is determined for steady-state motion. The stability of the equilibrium shape with respect to small perturbations of the axes of the ellipsoid is investigated through analysis of potential energy in the neighborhood of the extremum.

1. Lagrangian and Routh Functions. A gas bubble moves in an ideal fluid which is at rest at infinity. It is assumed the pressure p of the gas within the bubble is constant and is a function of bubble volume, p(V). Then the Lagrangian function determining the dynamics of the bubble is

$$L = T - \sigma S - p_\infty V + \int p(V)\,dV$$

Here, σ is the coefficient of surface tension of the fluid; S and V are the surface area and volume of the bubble. The kinetic energy T of the fluid is a quadratic form in generalized velocities,

$$T = \frac{1}{2} Mv^2 + v \sum_i M_i \dot{q}_i + \sum_i \sum_j \frac{1}{2} M_{ij} \dot{q}_i \dot{q}_j$$

Here v is the velocity of the translational displacement of the bubble (it is assumed for simplicity that only one component of the velocity vector is different from zero). The apparent masses M, M_i, and M_{ij} are functions of the generalized coordinates q_i determining the volume and shape of the bubble.

Because of the homogeneity of the space in the dynamic system, the law of conservation of momentum

$$P = \partial T / \partial v \tag{1.1}$$

is valid. The equations of motion for a bubble with a given momentum P can be obtained from the Routh function [4]

$$R = L - v P \tag{1.2}$$

The velocity v should be expressed through the momentum P and the generalized velocities q_i by means of Eq. (1.1) for the conservation of momentum.

The Routh function (1.2) is a Lagrangian function for the reduced system with local generalized coordinates q_i.

2. Potential Energy of the System. The second term appearing in the Routh function (1.2), taken with the opposite sign, does not depend on the generalized velocities $\dot{q}_i$, is the Routh potential, and plays the part of the potential energy for the reduced system.

The potential energy U is

$$U = P^2 / 2 M + \sigma S + p_\infty V - \int p(V)\,dV \tag{2.1}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 49-54, September-October, 1974. Original article submitted January 17, 1974.

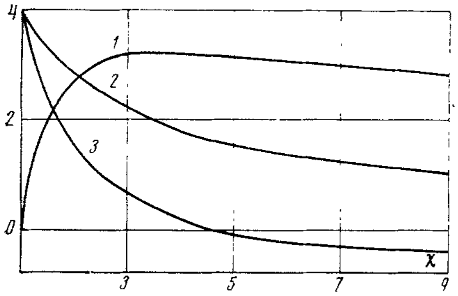

Fig. 1

Following [4], we designate steady-state motion as motion for which the coordinates determining the volume and shape of the bubble (local coordinates) remain constant. According to this definition, the motion of a bubble in a homogeneous flow of fluid is steady-state motion. Since steady-state motion corresponds to the equilibrium position of a reduced system with the Lagrangian (1.2), a necessary condition for the existence of such motion is expressed by equating the first variation of the potential energy to zero, $\delta U = 0$. If the system has an infinite number of degrees of freedom, the necessary condition is written as

$$\partial U / \partial q_i = 0$$

If the volume V is selected as one of the generalized coordinates, this condition yields an exact relation

$$(p_0 - p_\infty) V_0 = {}^2/_3\, \sigma S_0 - {}^1/_2 M v_0^2 \tag{2.2}$$

for the derivation of which it is necessary to consider that $M \sim V$, $S \sim V^{2/3}$.

Here and in the following, the subscript 0 denotes that the corresponding quantity refers to steady-state motion.

A sufficient condition for the stability of steady-state motion is the positive-definiteness of the second variation $\delta U > 0$, or the quadratic form of the second differential of the potential energy must be positive-definite for a system with a finite number of degrees of freedom.

The functions M and S appearing in Eq. (2.1) can be expressed through dimensionless quantities m and s which are independent of bubble volume,

$$M = {}^4/_3\, \pi\rho l^3 m, \quad S = {}^2/_3 \pi l^2 s, \quad V = {}^4/_3 \pi l^3 \tag{2.3}$$

Here l is the radius of a sphere having a volume equal to the volume of the bubble.

Let l_0 and m_0 be values of l and m in the equilibrium state; v_0, the velocity of steady-state motion of the bubble; and p_0, the gas pressure within the bubble in the equilibrium state when the bubble volume is V_0. Let z be a dimensionless parameter which determines the deviation of bubble volume from the equilibrium value:

$$l = l_0 (1 + z) \tag{2.4}$$

Substituting Eqs. (2.3) and (2.4) in Eq. (2.1) and omitting the constant dimensional factor ${}^4/_3 \pi\rho l_0^3 v_0^2$, one can then obtain an expression for the potential energy in terms of the dimensionless functions

$$U = (1 - 3z + 6z^2)\frac{m_0^2}{2m} + \frac{1}{W}\left[(1 + 2z + z^2)s + \pi_\infty (3z + 3z^2) - \pi_0 \left(3z + \frac{3}{2}(2 - 3\gamma) z^2\right)\right] \tag{2.5}$$

$$W = \frac{2 l_0 \rho v_0^2}{\sigma}, \qquad \pi_\infty = \frac{p_\infty 2 l_0}{\sigma}, \qquad \pi_0 = \frac{p_0 2 l_0}{\sigma}$$

$$\gamma = -\frac{V_0}{p_0} \frac{dp}{dV}\bigg|_{V = V_0}$$

Equation (2.5) for U is written with an accuracy to second order in small z, which is necessary for the investigation of stability. In the case of a polytropic process, γ agrees with the index of polytropy.

The functions m and s can be calculated explicitly for a bubble in the shape of a triaxial ellipsoid with semiaxes l_x, l_y, and l_z, the velocity v of which is directed parallel to the semiaxis l_z. A triaxial ellipsoid can be assigned by means of the parameter l, which defines the volume, and two other dimensionless parameters x and y,

$$\begin{gathered} l_x = \sqrt{a}\, l, \ l_y = \sqrt{b}\, l, \ l_z = \sqrt{c}\, l \\ a = (1 - x)^{-2/3} (1 - y)^{1/3}, \quad b = (1 - x)^{1/3} \ (1 - y)^{-2/3} \\ c = (1 - x)^{1/3} \ (1 - y)^{1/3} \end{gathered} \tag{2.6}$$

The functions m(x, y) and s(x, y) are [5]

$$m = I / (2 - J),\ s = 3\,(c + \varphi / \sqrt{c}) \tag{2.7}$$

$$I = \int_0^\infty \frac{d\lambda}{\sqrt{(a+\lambda)(b+\lambda)(c+\lambda)}},\quad \varphi = \int_0^1 \frac{(1 - xy\lambda^2)\,d\lambda}{\sqrt{(1 - x\lambda^2)(1 - y\lambda^2)}}$$

For an axisymmetric ellipsoid, $x = y$; the functions I and φ, and, consequently, m and s also, and the potential energy U are expressed in terms of elementary functions

$$I(x, x) = \frac{2}{x}\left(1 - \sqrt{\frac{1-x}{x}}\,\mathrm{arc\,tg}\sqrt{\frac{x}{1-x}}\right) \tag{2.8}$$

$$\varphi(x, x) = x + \frac{1-x}{2\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

In the case of a small deformation of the axisymmetric ellipsoid ($x \ll 1$)

$$m(x, x) = {}^1/_2\,(1 + {}^3/_5\,x + {}^{81}/_{175}\,x^2),\quad s(x, x) = 6\,(1 + {}^2/_{45}\,x^2) \tag{2.9}$$

3. Equilibrium Conditions. Equation (2.4) in conjunction with Eqs. (2.5) and (2.6) determines the dependence of potential energy U(x, y, z) on the parameters of a triaxial ellipsoid. Taking the parameters x, y, and z as generalized coordinates, one can obtain equations defining steady-state motion

$$\partial U / \partial x = 0,\quad \partial U / dy = 0,\quad \partial U / \partial z = 0 \ \text{ for } \ z = 0 \tag{3.1}$$

The first two equations determine the equilibrium shape as a function of the Weber number. Eliminating the Weber number W from these equations, one can find a curve determining a series of equilibrium shapes:

$$\frac{\partial s}{\partial x}\frac{\partial m}{\partial y} - \frac{\partial s}{\partial y}\frac{\partial m}{\partial x} = 0 \tag{3.2}$$

Equation (3.2) determines a series of axisymmetric equilibrium shapes when $x = y$. Numerical analysis of Eq. (3.2) shows that there are no other equilibrium shapes in addition to this series.

For an axisymmetric series of ellipsoids, Eqs. (3.1) make it possible to determine the dependence of the Weber number W and of $\Delta\pi = \pi_0 - \pi_\infty$ on the degree of deformation of the ellipsoid. Since for $x = y$ the partial derivatives with respect to x and y are half the total derivative with respect to x of the corresponding functions, these dependences can be determined with the help of Eq. (2.8) from

$$W = 2\frac{ds}{dx} \Big/ \frac{dm}{dx},\quad \Delta\pi = \frac{2}{3}s - \frac{1}{2}mW \tag{3.3}$$

The first relation has been determined [1-3] (curve 1, Fig. 1). The curve for the second dependence on the degree of deformation of the ellipsoid $\chi = l_x / l_z$ is given by curve 2, Fig. 1 [$3\Delta\pi(\chi)$].

There is interest in the relation $3\Delta\pi' = 6\,(p_0 - p_*)\,l/\sigma$ (p_* is the pressure at the stagnation point). From the Bernoulli integral it follows that $\Delta\pi' = \Delta\pi - {}^1\!/_2 W$. A curve for this relation is also shown in Fig. 1 (curve 3). The zero of the function $\Delta\pi'$ at the point $\chi = 4.83$ physically means that the curvature of the bubble surface at a point on the axis of rotation goes to zero and that the surface of the bubble at this point becomes concave for larger χ.

The corresponding two-dimensional problem has been considered [6]. The relations $W(\chi)$ and $\Delta\pi'(\chi)$ found in that work on the basis of an exact numerical solution are in qualitative agreement with Eqs. (3.3).

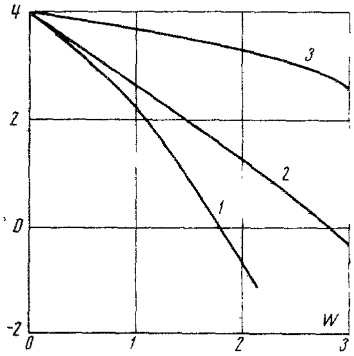

Fig. 2

For small deformations ($W \ll 1$), expansions with respect to small x or small W in Eqs. (3.3) can be found with the help of Eqs. (2.9),

$$W = {}^{32}/_9\,x\,(1 + {}^8/_{105}\,x) \tag{3.4}$$
$$\Delta\pi = 4 - {}^1/_4\,W - {}^9/_{320}\,W^2$$

4. Stability Conditions. By analyzing Eq. (2.5) one can determine a criterion for the stability of equilibrium shape with respect to small, not necessarily axisymmetric, perturbations of bubble shape and volume. These perturbations are specified by the deviations x, y, and z from the equilibrium state.

For a stable equilibrium state, it is sufficient that the potential energy U at the equilibrium point reach a minimum value, or

$$\frac{\partial^2 U}{\partial x^2}>0, \quad \begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\,\partial y} \\ \frac{\partial^2 U}{\partial x\,\partial y} & \frac{\partial^2 U}{\partial y^2} \end{vmatrix}>0, \quad \begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\,\partial y} & \frac{\partial^2 U}{\partial x\,\partial z} \\ \frac{\partial^2 U}{\partial x\,\partial y} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y\,\partial z} \\ \frac{\partial^2 U}{\partial x\,\partial z} & \frac{\partial^2 U}{\partial y\,\partial z} & \frac{\partial^2 U}{\partial z^2} \end{vmatrix}>0 \tag{4.1}$$

In expanding the determinants, one can use the fact that $x=y$, $z=0$ in the equilibrium state,

$$\frac{\partial^2 U}{\partial x^2}=\frac{\partial^2 U}{\partial y^2}, \quad \frac{\partial^2 U}{\partial x\,\partial z}=\frac{\partial^2 U}{\partial y\,\partial z}=\frac{1}{2}\frac{d}{dx}\frac{\partial U}{\partial z}, \quad \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial x\,\partial y}=\frac{1}{2}\frac{d^2 U}{dx^2}$$

The conditions (4.1) lead to

$$\frac{\partial^2 U}{\partial x^2}>0, \quad \left(\frac{\partial^2 U}{\partial x^2}-\frac{\partial^2 U}{\partial x\,\partial y}\right)\frac{d^2 U}{dx^2}>0 \tag{4.2}$$
$$\left(\frac{\partial^2 U}{\partial x^2}-\frac{\partial^2 U}{\partial x\,\partial y}\right)\left[\frac{\partial^2 U}{\partial z^2}\frac{d^2 U}{dx^2}-\left(\frac{d}{dx}\frac{\partial U}{\partial z}\right)^2\right]>0$$

The first two conditions are equivalent to the following:

$$\frac{d^2 U}{dx^2}>0, \quad \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x\,\partial y}\right)U>0 \tag{4.3}$$

When the conditions (4.3) are taken into consideration, the last condition in (4.2) takes the form

$$\frac{\partial^2 U}{\partial z^2}\frac{d^2 U}{dx^2}-\left(\frac{d}{dx}\frac{\partial U}{dz}\right)^2>0 \tag{4.4}$$

Simultaneous satisfaction of conditions (4.3) and (4.4) is equivalent to satisfaction of conditions (4.2).

The calculations are considerably simplified if one uses the following relations at the point $x=y$:

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x\,\partial y}\right)I=\frac{1}{4}\frac{d^2}{dx^2}I(x,x) \tag{4.5}$$
$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x\,\partial y}\right)\varphi=\frac{1}{2(1-x)}+\frac{1}{4}\frac{d^2\varphi(x,x)}{\partial x^2}$$

These relations can be proven by direct differentiation of the functions inside the integral sign in (2.7). The first relation in (4.5) can be proven on the basis that I and all partial derivatives of I with respect to x and y are homogeneous functions of a, b, and c.

The left sides of inequalities (4.3) and (4.4) are determined from Eqs. (2.5)-(2.7), and the problem is reduced to differentiation of the elementary functions (2.8) when Eqs. (4.5) are taken into account.

The first inequality in (4.3) is satisfied for all values of the Weber number W and denotes a condition of stability for steady-state motion of an ellipsoid of rotation with respect to axisymmetric perturbations at constant volume. This result was obtained previously [3].

Calculation shows that the second condition in (4.3) is also always satisfied. Satisfaction of both conditions in (4.3) means that any steady-state motion is stable with respect to perturbations of the axes of a triaxial ellipsoid at constant volume.

The last inequality (4.4) is a condition for the stability of steady-state motion with respect to small perturbations of volume and axisymmetric perturbation of shape. The condition for stability with respect to small perturbation of volume only reduces to the inequality

$$\partial^2 U / \partial z^2>0 \tag{4.6}$$

With the help of Eqs. (2.5)-(2.8), the conditions (4.4) and (4.6) are transformed to

$$\pi_\infty>\frac{1}{\gamma}f-\Delta\pi, \quad \pi_\infty>\frac{1}{\gamma}f_1-\Delta\pi \tag{4.7}$$
$$f_1=\frac{2}{9}s-mW, \quad f=f_1+W\left(\frac{5}{6}\frac{dm}{dx}\right)^2 \Big/ \frac{d^2 U}{dx^2}$$

These conditions determine a critical value for the fluid pressure at infinity below which stable steady-state motion does not exist. From Eqs. (2.7), (2.8), (3.4), and (4.7), one can find expansions of the functions f and f_1 in terms of small W with an accuracy to terms of order W^2,

$$f_1={}^4/_3-{}^1/_2\,W-{}^{51}/_{640}\,W^2, \quad f={}^4/_3-{}^1/_2\,W+{}^3/_{80}\,W^2 \tag{4.8}$$

The relations $3f_1(W)$, $3f(W)$, and $\Delta\pi(W)$ are shown in Fig. 2 (curves 1-3, respectively).

5. Model of a Spherical Bubble. The conditions for equilibrium and stability of steady-state motion can be analyzed with the help of a spherical model. In this case, the bubble has only one degree of freedom determining the volume of the bubble. One should set $s = 6$ and $m = 1/2$ in the appropriate equation (2.5) for U.

The potential function U(z) depends only on z.

The equilibrium condition determines the relation

$$\Delta\pi = 4 - {}^1\!/_4\, W \tag{5.1}$$

The stability condition $d^2U/dz^2 > 0$ takes the form

$$\pi_\infty > \frac{1}{\gamma} f - \Delta\pi, \quad f = {}^4\!/_3 - {}^1\!/_2 W \tag{5.2}$$

A comparison of Eqs. (5.1) and (3.4) and of Eqs. (5.2) and (4.8) shows that the spherical model gives the correct asymptotic dependence of $\Delta\pi(W)$ and $f(W)$ for small Weber numbers. When $W = 0$, Eqs. (5.1) and (5.2) give exact relations for stable equilibrium of a spherical bubble acted on by the forces of gas and fluid pressure and by surface-tension force,

$$p_0 = p_\infty = \frac{2\sigma}{l}, \qquad l > \frac{\sigma}{p_\infty}\left(\frac{2}{3\gamma} - 2\right)$$

LITERATURE CITED

1. O. M. Kiselev, "Determination of gas-bubble shape in axisymmetric flow of fluid," Zh. Prikl. Mekhan. Tekh. Fiz., No. 3 (1963).
2. D. W. Moore, "The velocity of rise of distorted gas bubbles in a liquid of small viscosity," J. Fluid Mech., 23, No. 4 (1965).
3. O. V. Voinov, A. M. Golovin, and A. G. Petrov, "Motion of ellipsoidal bubble in fluid of low viscosity," Zh. Prikl. Mekhan. Tekh. Fiz., No. 3 (1973).
4. F. R. Gantmakher, Lectures on Analytic Mechanics [in Russian], Nauka, Moscow (1966).
5. N. E. Kochin, I. A. Kubel' and N. V. Roze, Theoretical Hydrodynamics [in Russian], Pt. 1, Fizmatgiz, Moscow (1963).
6. O. M. Kiselev, "Problem of a gas bubble in plane flow of an ideal fluid," Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, No. 4 (1969).